

**Tough to Teach/Tough to Learn: Research Basis,
Framework, and Principles for a Practical Application
of TI-Nspire™ Technology in the *Math Nspired* series**

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Executive Summary

The *Math Nspired* series of curriculum supplements currently include *Algebra Nspired* and *Geometry Nspired*. The supplements grew from research on the “**tough to teach/tough to learn**” topics, which our item analysis of state tests showed to be common points of difficulty for many students. To determine the underlying reasons for the difficulty, we consulted the research on the reasons why students struggle with key concepts in Algebra and Geometry. We found that the source of difficulty is mathematical tasks with **high cognitive demand**, which use **schematic and strategic knowledge** of math. We concluded that the need is for learning activities which engage students in high cognitive demand mathematical tasks that require schematic and strategic knowledge of math. Each activity features:

- An **important topic** in the curriculum
- Tasks of **high cognitive demand**
- A technology-based exploration linking mathematical **actions** and **consequences**
- Questions to stimulate **reflection** and **sense-making**

The activities are carefully designed to have high mathematical fidelity, high cognitive fidelity, and high pedagogical fidelity.

At the heart of every *Math Nspired* learning activity is a special kind of document for TI-Nspire which is designed to **get students to the math faster**, by placing them in exploratory “microworlds” where they can easily and intuitively take mathematical **actions** on high-fidelity mathematical objects, observe the mathematical **consequences** of their actions, and then **reflect** on what they do. It is this attention to the action-consequence-reflection and sense-making structure that distinguishes these learning activities.

The whole system is designed to be used just as conveniently by new teachers as by their experienced colleagues. Accompanying lesson plans and student worksheets can be used “as is” or adapted by teachers to their students. Supporting web-based resources provide relevant professional development on pedagogical content knowledge, pedagogy, and technology.

Introduction

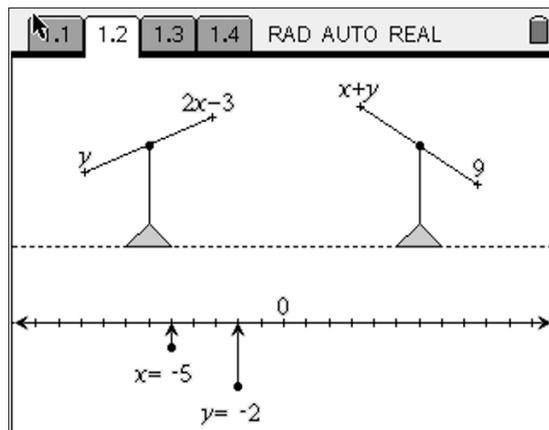
Mrs. Gonzales was reviewing the results from last year's Ohio state test in Algebra. She noticed that 46% of her students got this question correct, but 40% left it blank:

Mrs. Foyle told Yolanda that her test had 38 problems worth a total of 100 points. Each test problem is worth either 5 points or 2 points. Yolanda wanted to determine how many 2 point and how many 5-point questions are on the test. In your answer document, determine how many questions of each point-value are on the test.

She thought back to how she had carefully and deliberately taught students how to solve systems using substitution, giving examples, explaining how the steps proceed, and having students check their solutions. Her textbook included the substitution method, and she had also used her TI-Nspire handheld calculators to discuss solution by graphing. Yet she had students who would solve a system of equations for one of the variables and stop. And other students who would solve for x in one of the two equations and substitute that value into the same equation producing an identity, which they may then manipulate to the form $0=0$ and pronounce that $x = 0$. She saw the need to try something different this year.

She scanned the list of *Algebra Nspired* learning activities she had just downloaded. She found an activity called ***Balanced Systems*** that looked like it might provide another way of teaching this. In this TI-Nspire electronic document, students are presented with a number line and two movable pointers to control the value of x and of y . Two balance scales are shown, each representing a linear equation (a linear expression in x and y is on each side of each balance). The current values of x and y result in a visual display of the relationship between the evaluated expressions (the greater the value, the “heavier” that side of the scale). Students can try to find the values of x and y that make either or both of the equations balanced. (See Figure 1 below.)

Figure 1. Balanced Systems



When she tried the lesson in class, Mrs. Gonzales found that she was able to pursue sense-making questions that went beyond the procedural “What step comes next?” norm. While using the interactive electronic document in class, she asked the following questions of her students:

- “Which has the larger value in the scale on the right? How do you know?”
- “What does it mean when one of the scales is balanced?”
- “How many values of x and y will you be able to find that balance the scale on the left? That balance the scale on the right?”
- “What does it mean if you find values for x and y such that both scales balance?”
- “How would you write a symbolic representation of a balanced system, in which both scales were balanced?”
- “What are some strategies you might use to try to balance both scales? Why do you think these might give you an answer?”
- “Will you be able to find more than one ordered pair (x, y) that balances both of the segments? Why or why not?”
- “Will only pairs of integers balance a scale? Both scales? Explain your reasoning.”

Her students made conjectures about strategies for balancing the two scales simultaneously, such as “Just keep moving both points until you get one to balance, then move one of the points so the other segment gets more balanced and then go back to the first segment and move the other point so that it is balanced again”

The next two pages in the *Balanced Systems* activity (Figures 2 and 3) had students investigate whether it is possible to have more than one ordered pair (x, y) that will make both scales balance or if there always has to be one ordered pair (x, y) such that they will both balance. Suggested questions that drive student thinking included, “How are the two equations on page 1.3 and 1.4 like the two equations on page 1.2? How are they different?”

Figure 2. Balanced Systems (an inconsistent system)

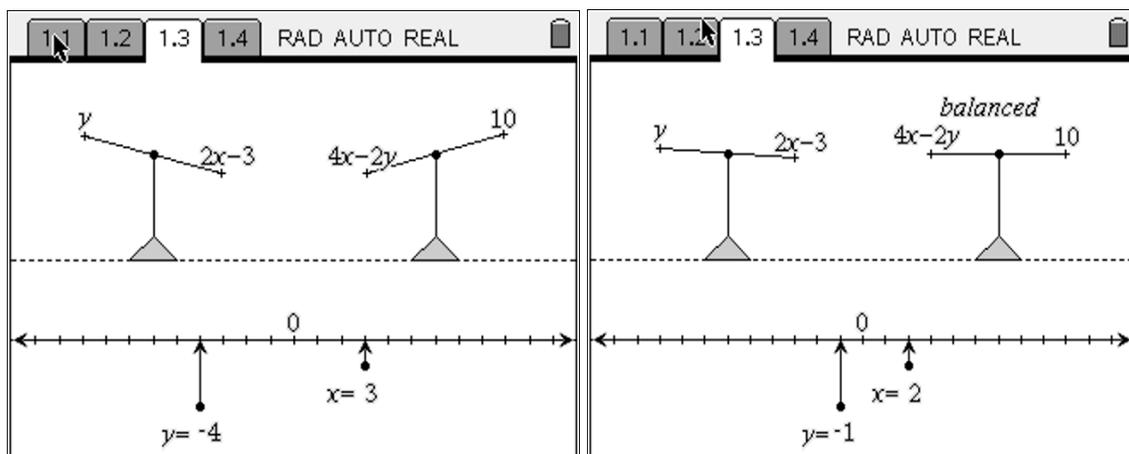
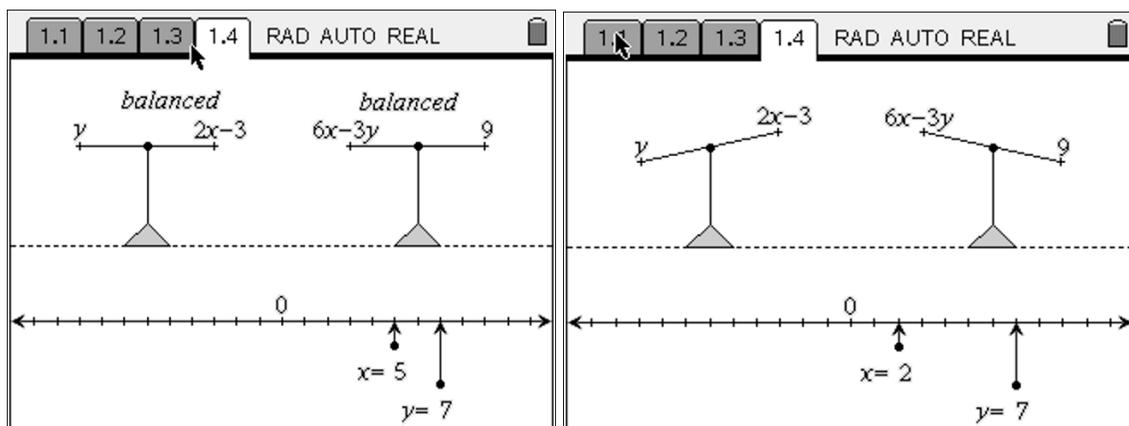


Figure 3. Balanced Systems (a dependent system)



Ms. Gonzales could immediately see the students dealing with types of questions previous years' classes didn't. She could see them building a deeper understanding of what a system of equations is. She was optimistic about how this year's students would do on questions using these concepts.

Mrs. Gonzales is not alone. Real improvement in the student success rate in Algebra is one of the biggest challenges in secondary math education. With more than a decade of experience with universal proficiency testing in K-12 mathematics achievement, it has become apparent that many (perhaps most) students are not prepared to succeed on the tests. Research on questions as to why the transition from arithmetic to algebra and geometry is often so difficult for students can provide valuable insights that have implications for instructional strategies (Kieran 2006). However, the development of practical technological tools is needed to help teachers of these subjects exploit those strategies. Developing the tools and the supporting learning activities for students requires understanding the underlying curricular issues in the mathematics and learning issues for students. If the approach is to be scalable, the tools and strategies must be accessible by typical Algebra 1 and Geometry teachers, not just by the "teacher leaders."

We believe the integration of state-of-the-art technology, both handheld and web-based, has the potential to aid greatly in both the effectiveness and the scalability of the solution. The sections that follow lay out 1) the research basis for the *Algebra* and *Geometry Nspired* programs, 2) the framework and design principles guiding the development of the programs, and 3) an overview of included resources. Further product information is available from the *Math Nspired* section of education.ti.com.

1. Research Basis for Algebra and Geometry Nspired

What do state assessments say about “Tough to Teach/Tough to Learn” mathematics?

We began our research by leading a team of researchers in an examination of performance results on Algebra and Geometry assessment tests across the fifty states. The goal was to identify particularly the “tough to teach/tough to learn” topics or concepts with which the majority of students often have difficulty on these tests. The content of items on particular state tests - Texas, Ohio, Florida, New Hampshire, Maine - for which an item analysis was readily available - provided additional insights. Items for which the p-value (percent correct) was less than 50% were identified and categorized according to content area and topic as well as cognitive demand.

Across the tests, certain content topics in Algebra emerged as problematic for students. These included:

- percentage/proportional reasoning,
- reasoning with algebraic concepts,
- rate of change/linearity/related graphs,
- systems of equations,
- manipulating expressions,
- exponential growth and decay.

Note that not all of these topics were on all of the tests, but when they were, the average number correct was below 50% across most of these state tests.

Insights into tough to teach/tough to learn geometry topics were gleaned from looking at the item analyses of the state tests from Texas, Florida, Ohio, Minnesota, Massachusetts, and New Hampshire. (These are the only states where we were easily able to find item analyses - Connecticut has some open ended items with low scores in these areas as well.). Topic areas where at least 40% of the students struggled with state test items were:

- Area, particularly when the tasks involved putting ideas together

- Volume
- Applications of the Pythagorean theorem in related situations
- Reasoning about geometrical ideas
- Transformations (usually in coordinate grid settings)
- Spatial/cross sections - related to 3 dimensional visualization
- Coordinate geometry settings (involve area, midpoints, circles, etc.)
- angles in polygons

To a lesser degree, items on arcs and angles in circles, similarity of figures, and right triangle trigonometry also posed difficulty.

The next step was to consult the available research on these topics, and conduct an analysis of the underlying causes of difficulty with each of these topics. As an example, we will consider the results of one such analysis, that of linear systems.

The Case of Linear Systems:

What does research say about why students struggle?

Let's examine the problem of linear systems of equations, and consider the examples in Figure 4 from a ninth grade end of course Florida Comprehensive Assessment Test and a ninth grade Texas Assessment of Knowledge and Skills:

Figure 4. Released assessment items from Florida and Texas ninth grade tests

In the system of equations $4x + 2y = 10$ and $3x + 7y = -18$, which expression can be correctly substituted for y in the equation $3x + 7y = -18$?

- A. $10 - 2x$
- B. $10 + 2x$
- C. $5 - 2x$
- D. $5 + 2x$

(Texas, 2004; 43% correct)

A local bakery is baking cakes for a restaurant owner. The bakery sells one kind of cake for \$16 and another kind of cake for \$12. The restaurant owner will pay \$1000 all together for 70 cakes. This information can be represented by the following equations, where x represents the number of \$16 cakes and y represents the number of \$12 cakes.

$$16x + 12y = 1,000$$

$$x + y = 70$$

How many \$16 cakes should the bakery bake for the restaurant owner?

(Florida, 2006; 38% correct)

These and similar results from other state assessments suggest that students across the United States struggle with systems of linear equations (Burrill and Dick 2008).

We expect that very few algebra teachers, if any, would suggest that the assessment examples above were unreasonably difficult or required computations that were overly complicated. Indeed, our experience in discussing such items with teachers has been overwhelming agreement that the kinds of understandings necessary to successfully answer these items should be expected of the vast majority of our students.

A closer examination of the items reveals some possible clues to the apparent difficulty. Note that the items ask for representational or contextual interpretations of the solutions or of particular steps in the solution process. Yet, when we ask teachers about how they teach systems of linear equations in their algebra curricula, the discussion almost always centers on solution methods (substitution, elimination or “addition-subtraction” method, graphical) and types of systems: dependent (one line); independent (two intersecting lines); and inconsistent (two parallel lines). Contrast this with the richness of the concept map of systems of equations proposed by Proulx, et al (2009), which includes:

- *representing* (technology, context, graph, equations, table of values),
- *conceptualizing* (“What is a system? – combination of constraints, – satisfying both equations; “What represents a relevant situation? – relevance of the tool, – model and solve problems; “What are the constraints?”),
- *solving* (table of values, graphically, algebraic methods, context), and
- *interpreting* (possibilities/numbers/meaning of solutions, graphically, algebraically, domains/ restrictions/ context dependent, continuous vs. discrete, number of equations vs. number of unknowns).

In short, learning the mechanics of solving a given system is important but only a part of building a robust understanding of linear systems.

Researchers raise additional issues that emerge as problematic in developing students' understanding of systems of equations. For example, Means (2008) suggests two common uses of letters are:

(1) to specify a single unknown or bounded set of unknowns, such as $4x = 8$, where $x = 2$, and

(2) to specify an infinite set of possible unknowns, such as $4x = 8 + y$, for which there is an infinite set of (x,y) pairs that satisfy the equation.

“Unfortunately for students, these different uses are typically left implicit.” (page 5)

We would add that neither of these uses is incompatible with the idea of a letter representing a variable, in that an equation is a statement that takes on a “true” or “false” value depending on the numerical value(s) substituted for the variables that appear.

“Finding the unknown” is the same as “finding the value that makes the equation true.”

Students also may adopt a preferred method regardless of the nature of the two equations. This focus on a single procedure can hinder the development of any flexibility to give meaning to equations and their solution (De Lima and Tall 2006), with students reluctant to shift from procedural methods to more flexible manipulation of the symbols.

De Lima and Tall also suggest that one of the barriers to learning can be “met–before,” concepts students have met early in their mathematical work. “Some met–before—such as those in a well designed curriculum—can be a positive foundation for successful development of concepts. Others, such as epistemological obstacles studied by the French School (Brousseau 1997), can cause conflict in a new context and have a negative effect on learning.”

One met–before that may pose difficulty for students working with systems of equations is a view of the equals sign as a signal to “do something” (Kieran 1981) rather than a statement of the relationship between two mathematical expressions. For example, elementary teachers recognize that many students say “14” in response to the question: “What goes in the blank? $6 + 8 = \underline{\quad} + 3$.” Students learning algebra who see an equation as a mathematical calculation are challenged by the appearance of two equations. They use misplaced rules, do not mention equivalence between the two

expressions that make up the equation, and often do not seem to use the procedure of applying the same operation to both sides to simplify the equation as part of the process of finding a solution (Tall 2004). Thus, their procedural knowledge for solving linear equations in one variable is limited, fragile, and prone to error, understandably inhibiting their success in transferring this knowledge to solving a system of equations.

We believe technology has an important role to play in addressing barriers to understanding such as the example above. The next section examines this issue.

Multiple roles of technology in mathematics teaching

“Technology is essential in teaching and learning mathematics. It influences the mathematics that is taught and enhances student learning.”(NCTM 2001).

How is technology used in our classrooms? Does NCTM have it right or is technology really just a crutch that takes away student responsibility for learning? Graphing calculators, spreadsheets, and computer algebra systems can all be thought of as powerful computational “toolkits.” Too often, however, such technology has been relegated to the role of performing a task that is too cumbersome to do by hand – or to apply a concept in a “real” situation where the numbers are not “nice”. For example, much use of technology, including graphing calculators, occurs after the content has been taught, where the technology is used to affirm or to practice a concept or to apply the concept in a context with “real” numbers. This approach reflects the teachers' view of their role as delivering content and of their stance toward the calculators as tools to make things easier.

We suggest that this view of the technology as computational aid dominates most debates over the appropriateness of technology use in mathematics education. It is one lens through which technology use may be viewed. For proponents with this lens, the gains to be made are in the extended reach of important mathematical problems afforded to students. In terms of the NCTM vision, the complex calculation tasks that can be handed off to technology truly “influences the mathematics that is taught.” Kennedy

(1995) makes reference to a “tree of mathematics,” and how technology has allowed more students to climb the tree, reaching higher branches than students of the past. For opponents of technology with this lens (and we believe virtually all opponents tend to view technology through this lens), the central issue is that tasks are being delegated to technology that simply should not be delegated at all— the availability of a willing servant breeds laziness and dependence, for even simple computational tasks. Note that there might well be a ready acknowledgment that there are tasks for which the technology is the only reasonable option, but for opponents the risks of dependence far outweigh the benefits. Usiskin (1978) referred to the “crutch premise” concern many years ago in his discussion of the controversy over simple arithmetic calculators. We are clearly hearing echoes of that crutch premise in the current debates, especially for computer algebra systems.

There is, however, a very different lens to consider: that technology can play a critical role on the road to mathematical reasoning and sense-making by providing settings in which students explore core mathematical concepts and encounter the need for justification of a conjecture or generalization. This use of technology in setting up environments for exploration is quite different from the use of technology as simply a computational task performance aid. For example, the teacher who chooses to have students explore the effect of changes in the values of a , b , and c in $y = ax^2 + bx + c$ and of a , h , and k in $y = a(x - h)^2 + k$, then making generalizations about the two forms, would seem to view her role as providing situations for students to develop their own understanding under her guidance, where the technology is a key tool in setting up the situation.

Too often no real distinction is made between the two lenses. Belfort & Guimares’ (2004) careful analysis of technology-based materials for mathematics revealed the following broad categories of serious shortcomings:

1. the author’s interest is on mastering the use of the technology where the mathematics is secondary;
2. the activity is merely a demonstration of an idea where students are treated as

- spectators;
3. the activity revisits a mathematical topic to show how it can be done in a simple way with the new technology where the students' role is verification;
 4. the author replicates activities from the point of current instructional materials, underestimating the technology's potential, where the ideas are fragmented and obtaining a formula is often the objective.

We believe that many, if not most, of these shortcomings are the result of viewing the role of technology through the first lens. The opponents of technology are correct when they object to applications which exemplify these shortcomings, but we argue that these shortcomings are due to the author's basic misunderstanding of the role of technology in teaching, not the use of technology itself.

With this review of the research basis upon which we draw, we are now ready to show how we have derived from the research a framework and design principles for the *Math Nspired* products using TI-Nspire technology.

2. Framework and principles for guiding use of TI-Nspire

The design of the TI-Nspire solution draws on the research surrounding the causes of difficulty in learning mathematics and the research on the role of technology in mathematics teaching. What has emerged is a framework and principles that can guide the instructional uses of TI-Nspire in ways that exploit its potential to be an integral aid to improving the learning and teaching of Algebra and Geometry – not a sophisticated crutch. The goal is to outline a set of principles for the essential role of technology that is consistent with the “tough to teach/tough to learn” analysis, and which will guide curriculum developers and teachers toward new opportunities for student learning made possible by the technology of TI-Nspire.

The principles are decidedly not technology- or pedagogy-neutral; they are prescriptive design principles for an instructional strategy using TI-Nspire. They reflect a set of values and assumptions regarding the needs of students and teachers to improve their mathematics learning and teaching, with a clear eye toward exploiting the special affordances of the TI-Nspire technology. The principles make explicit suggestions regarding *which* content should be targeted, *what* kind of activities students should engage in around that content, and *how* and *where* in the instructional strategy TI-Nspire should be used in supporting or facilitating those activities. The principles guide curriculum content authors in producing both electronic and hard copy materials which combine to provide practical solutions to these important tough to teach/tough to learn topics. Teachers will adapt and use the materials best when they understand these principles.

There are four principles, all based on a conceptual framework of mathematics teaching and learning:

1. Curricular importance
2. Cognitive demand
3. Action-consequence
4. Reflection and sense-making

We will elaborate on each of these principles below, and examine the conceptual framework upon which they are based.

Principle 1: Curricular Importance

Choose topics of fundamental importance in school mathematics curricula.

As obvious as this principle may sound, in practice there is often a tendency to “show off” new technology, as discussed in the previous section, by applying it to novel or complex topics that at best are on the periphery of the curriculum or that might be readily accessible by only more sophisticated students. Advanced or novel technology need not be restricted to advanced or novel topics (doing so only creates the perception that technology is not appropriate at the basic level). Rather, one should aim for activities that provide *new insights* and *deeper understanding* of fundamental topics (such as those identified in the “tough to teach/tough to learn” analysis, described above). Teachers who may not have embraced graphing calculator technology will quickly see these improved ways of helping students to engage with those major ideas that appear over and over again at different grade levels throughout the mathematics curricula.

We believe that identification of “big idea” strands which systematically develop core knowledge will provide a more coherent approach to curriculum development, while minimizing (or correcting) misconceptions and “met-before” problems. This applies equally to whole textbooks and to curriculum supplements.

Principle 2: Cognitive Demand

Activities should include inquiry tasks of high cognitive demand.

We can begin by distinguishing two very broad categories of tasks in mathematical activities : *performance* and *inquiry*. By *performance* we mean the recall or recording of information, the creation of mathematical objects, direct actions taken on mathematical objects, and procedures (algorithmic sequences of actions). Examples of performance activities include geometric constructions, numerical computations, algebraic manipulations, transformations, measurements, translations between notation systems, graphing, diagramming, displaying, collecting, sorting, etc. By *inquiry* we mean the mathematical sense-making that can only result from purposeful reflection on the part of the student: finding and describing patterns (inductive reasoning), conjecturing, generalizing, abstracting, connecting between representations, deducting, predicting,

testing, relating, justifying, proving, and refuting. Both performance and inquiry tasks can be found in mathematical activities at every level. Thus, *the intent here is NOT to cast inquiry tasks as “good” and performance tasks as “bad.” Many good mathematical activities will include a mix of both kinds of tasks. However, a mathematical activity which is implemented in a way that emphasizes only performance tasks to the exclusion of inquiry tasks provides little opportunity for students (and teachers) to make sense of the mathematics.*

Relating Cognitive Demand to Mathematical Tasks

Making a distinction between performance and inquiry activity corresponds well to the distinction between low and high cognitive demand tasks as described in the Mathematical Tasks Framework (Stein, Smith et al. 2000). An organizing principle of the Mathematical Tasks Framework is the cognitive demand of a task. Moreover, the cognitive demand of a particular task is not a static attribute—the level of cognitive demand of a task can shift as students work on it and gain experience with it, and teachers can have great influence on this shift of level – facilitating or inhibiting it. Stein et al. identify three phases through which tasks unfold: 1) as they appear in curricular and instructional materials, 2) as they are posed by the teacher, and 3) as they are implemented by students in the classroom.

The teacher can change the level of cognitive demand not only at phase two but also at phase three through the type of assistance or direction provided as feedback to students as they work. These changes, in turn, have consequences ultimately in student learning outcomes.

The Mathematical Tasks Framework identifies two lower level categories of tasks: *memorization* and *procedures without connections*, and two higher level categories: *procedures with connections* and *doing mathematics*. Here are descriptions of each level as described by Stein et al. (*op.cit.*)

Memorization. Memorization tasks involve simply reproducing previously learned facts, rules, formulae, or definitions (or committing these to memory). These tasks can be performed without making any connections to underlying concepts or meanings.

Procedures without connections. These are algorithmic tasks that are focused on producing correct answers. There is no ambiguity in what steps need to be performed and the procedural task can be successfully completed without making any connections to underlying concepts or meanings.

Procedures with connections. These tasks involve procedures, but students need to engage with the underlying concepts and meanings in order to successfully complete the task. They often involve multiple representations and require making connections. These tasks are intended to develop deeper understanding of the underlying concepts and meanings.

Doing mathematics. These tasks require complex, nonalgorithmic thinking. There is not a predictable well-rehearsed path suggested by the task, instructions, or by previously worked example. These tasks require students to explore and understand the nature of mathematical concepts, processes, or relationships and to analyze and actively examine task constraints. They may involve some level of anxiety or frustration for the student due to the unpredictability of the solution process.

Relating knowledge types to cognitive demand level

In their work on science education assessment items, Li and Shavelson (2003) categorize the knowledge on which students draw in four distinct types and characterize the uses of each type of knowledge:

Figure 5: Types and Uses of Knowledge

<i>Knowledge Type</i>	<i>Use of Knowledge</i>
Declarative	Defining, providing an example
Procedural	Executing and performing procedures
Schematic	Explaining, justifying, predicting, hypothesizing
Strategic	Choosing knowledge to use, formulating strategies, raising questions, defining problems, heuristics

The correspondence between these four knowledge types and the four cognitive demand categories of the Stein, et al Mathematical Task Framework is not an exact match, but striking, nevertheless.

- Declarative knowledge is built from memorization of facts, definitions, principles, etc. and includes identifying and constructing examples.
- Procedural knowledge, as described by Li and Shavelson, is based entirely on the rote mechanics of carrying out a step-by-step algorithm, exactly the knowledge necessary to successfully complete a procedure without connections task, which also requires use of declarative knowledge. Of course, procedural knowledge would also be required to carry out a procedure with connections task.
- Schematic knowledge grows out of noticing and describing patterns and connections, and making predictions based on patterns and relationships, including those between representations. Hence, schematic knowledge comes into play with the higher cognitive demand task categories of procedures with connections as well as doing mathematics. Think of “schematic” as relating to the “larger scheme of things.” That is, how does one mathematical idea compare/contrast/relate to other mathematical ideas? In any case, it is the key idea behind “schematic” knowledge that really needs to be communicated: it is the rich knowledge of mathematical relationships, connections, patterns, representations, and models.
- Strategic knowledge includes skills with problem solving heuristics and the “management” of one’s own knowledge resources (what some call *metacognitive* knowledge). Thus, strategic knowledge is critical for the kind of problem solving included in the task category doing mathematics.

Of course, a single mathematical task could require all four kinds of knowledge (a doing mathematics task might require one to remember a specific definition, an example of declarative knowledge). The following figure suggests how the knowledge and task frameworks relate to each other.

Figure 6: Relating Knowledge Type to Task Type

TASK CATEGORY		KNOWLEDGE USE THAT MAY BE REQUIRED			
		<i>Declarative</i>	<i>Procedural</i>	<i>Schematic</i>	<i>Strategic</i>
<i>High Cognitive Demand</i>	<i>Doing mathematics</i>	✓	✓	✓	✓
	<i>Procedures with connections</i>	✓	✓	✓	
<i>Low Cognitive Demand</i>	<i>Procedures without connections</i>	✓	✓		
	<i>Memorization</i>	✓			

Principle 3: Action-Consequence

A TI-Nspire document should provide an environment for students to deliberately take mathematically meaningful actions on objects and to immediately see the mathematically meaningful consequences of those actions.

Dick (2008) posed the following pedagogical “axiom” in a discussion aimed at the designers of technology tools for mathematics education:

Students learn mathematics by taking mathematical actions (e.g., transforming, representing, manipulating) on mathematical objects (e.g., symbolic expressions, graphs, geometrical figures, physical models), observing the mathematical consequences of those actions, and reflecting on their meanings. Students’ reflections on mathematical consequences of mathematical actions on mathematical objects are the fuel for feeding the cycle of prediction–conjecture–testing that ultimately leads to proofs or refutations. (p. 334)

A measure of a technological tool’s faithfulness to this pedagogical principle lies in how easily and clearly it is perceived to:

- (a) faithfully represent mathematical objects and their behavior,

(b) allow mathematical actions on those objects, and

(c) provide clear evidence of the mathematical consequences of those actions.

To apply the action-consequence principle with inquiry resources created for TI-Nspire, we defined three major design goals:

- Transparency
- Visual Proximity and Immediacy
- Dynamic Links

Transparency. Heid (1997) uses the term *transparency* to refer to another construct related to the efficacy of technology: “The degree of transparency is the extent to which the technology being used highlights the mathematics that is being studied rather than obscures it.” (page 6). Using TI-Nspire technology, we can create a new resource for mathematical inquiry which allows the student new opportunities to take mathematically meaningful actions and provides more direct access to their mathematically meaningful consequences. The design goal is to help the student ***interact with the MATH, NOT the technology!*** In other words, the first action-consequence design goal is:

- ***Students working on TI-Nspire should feel that they are interacting with mathematical objects directly – and not interacting with the operating system of a machine.***

For example, the direct manipulation of a geometric object on screen can result in corresponding changes in the numerical measurements of some important characteristics such as the area and perimeter. In turn, the values of these numerical measurements might be graphed (one as a function of the other), or be used as parameters controlling the characteristics of some other mathematical object.

Ideally, an action-consequence document is virtually “menu free” in the sense that the student can engage in the mathematics immediately at the level of the screen (Kaput 1992). Contrast this with the more familiar type of activities that adopt the “toolkit” perspective of technology, by calling on the student to perform operations on the

calculator. Such activities are still valuable, for they can provide examples of the extended reach of mathematical problem solving afforded to students. However, such activities also have a greater tendency to lapse into directions on keystrokes and menus in lieu of the mathematics.

Visual Proximity and Immediacy is the second design goal. The potential value of action-consequence linkages in providing compelling learning experiences for students depends on how directly accessible and visually obvious the results are. The *visual proximity* and temporal *immediacy* of the results of an action itself are attributes of the technology that can aid the student in making connections by making them more obvious and by reducing cognitive load. In general, the closer in space and time the consequences are to the original actions as perceived by the user, the better – ideally on a single screen (page). Thus, the goal is to ***make the math IMMEDIATE and VISIBLE!***

- ***Students working on TI-Nspire should immediately see the mathematically meaningful consequences of the deliberate and mathematically meaningful actions they perform.***

Use of Dynamic Links is the third design goal. A *dynamic link* connects two or more objects (perhaps across representations) so that changes in one are immediately reflected in the others. In math instruction, these dynamic links provide:

- settings for mathematical exploration
- immediate visual consequences
- opportunities for prediction (What would happen if I made this change? What change should I make to get a specific result?)

Conventional representations of relationships, such as placing static graphs and equations side by side on a page, cannot adequately represent the way two mathematical objects continuously vary in an action-consequence relationship. Even switching between a graphical view and an equation view on a conventional graphing calculator obscures the connection. A dynamic link is needed to fully represent the relationship between a mathematical action and its related consequences. Thus, the design goal should be to **link the math dynamically!**

- *provide two or more external representations linked together in such a way that the actions performed in one representation have virtually simultaneous discernible consequences in the others.*

It is also helpful to contrast the use of the TI-Nspire document for the action-consequence principle with other, more static uses of the document. Teachers new to TI-Nspire often view its document model as simply as a nice organizational feature (for example, with presentation potential similar to Powerpoint, or for creating and distributing self-contained electronic worksheets for students). However, a TI-Nspire action-consequence document – an electronic document equipped with a compelling dynamic link – shares many of the characteristics of a software *microworld* - a learning environment in which a student can engage with mathematical objects having specific action-consequence rules under well-defined constraints (Kaput 1992; Balacheff and Kaput 1996). Indeed, TI-Nspire could be viewed as a “microworld-maker.”

What mathematical objects are dynamically linked?

Action-consequence documents are based on at least two dynamically linked mathematical objects:

- a “driver” object – a mathematical object that the student can take an action on by directly manipulating or editing, and
- a “driven” object – the resulting visible consequence to the action

Typical mathematical objects include a point location, expression, numerical value (parameter or variable value), graph of a function, or a geometric object (segment, polygon, circle, vector, etc.). Any of these could be dynamically linked, as shown in Figure 7, below.

Figure 7: Options for dynamically linking mathematical objects in an action-consequence document

Action (Driver)		Consequence (Driven)
Point location (in the plane or on a number line)	- - -	Point location (in the plane or on a number line)
Expression		Expression
Numerical value (parameter or variable value)	→	Numerical value (parameter or variable value)
Graph of a function	→	Graph of a function
Geometric object (segment, polygon, circle, vector, etc.) or associated attribute or associated measurement		Geometric object (segment, polygon, circle, vector, etc.) or associated attribute or associated measurement

The solid arrow in Figure 7 indicates an example. Graphing a linear function $y = mx + b$ in the usual Graphs & Geometry environment on TI-Nspire provides a two-way action-consequence environment, in which the driver is the graph: rotating the graph of the line produces a resulting change in the (driven) numerical value of the slope in the equation for the line. Similarly, translating the graph results in a change in the y -intercept. Conversely, the driver can be the expression: editing the equation for the line yields an immediate change in the graph. The dashed arrow indicates another example, one of the simplest versions of an action-consequence document: the student moves (drives) one point and observes how the other (driven) point changes position; the student is challenged to find the rule which links all the instances of the ordered pairs of points. Principle 3 focuses specifically on the technology-based component of Algebra Nspired, the action-consequence document. The fourth design principle is about the whole

learning activity, including classroom dialog.

Principle 4: Reflection and sense-making

Learning activities built around TI-Nspire action-consequence documents must explicitly promote student reflection – especially the posing of questions for sense-making and reasoning (including explanation and justification).

The new NCTM Secondary Curriculum Focal Points are founded entirely on two simple guiding goals: *sense-making* and *reasoning*. In authoring a learning activity based on a TI-Nspire action-consequence document, the emphasis must be on making the mathematics meaningful! No matter how powerful and compelling the dynamic link may be in an action-consequence document, the key to unlock student learning lies in the interactions and reflections about them: the tasks, questions posed and feedback supplied by the teacher (and/or in any accompanying student worksheet and/or in peer-to-peer discussion while students are engaged in the activity), intended to get the students to think deeply about the knowledge they are building.

Formulating good questions

Even the best math teachers find it difficult to formulate consistently good questions (Silver, Mesa et al. 2009). If all the actions are entirely directed by the teacher (or by the paper handout or by the instructions in the TI-Nspire document itself) and the student is only asked to record consequences, then the learning opportunities will fall far short of the potential available when students have adequate opportunities to conjecture, reflect, explain, and justify (in short, when students engage in *inquiry mathematics*).

Black and colleagues (Black, Harrison et al. 2004), writing on formative assessment, suggest that there are only two reasons to ask a mathematical question in instruction: either 1) to *evoke* students' mathematical thinking, or 2) to *explore* how students are thinking mathematically for the purpose of informing instruction. Similarly, Silver, Kilpatrick, and Schlesinger (1990) talk about the importance of teachers having the opportunities to engage students in “dialogues that provoke or probe.” With this in mind, we offer some general types of questions for teachers to keep in mind as they enact activities by using an action-consequence document. These are examples of questions

that push students to make sense of mathematics and to reason about mathematical concepts.

- **Compare and contrast:**

- “How are alike? How different?”

- **Given a condition, predict forward:**

- “What would happen if . . . ?”

- **Given an outcome, predict backward:**

- “How can I make ... happen?” “Is it possible to ... ?”

- **Analyze a connection/relationship:**

- “When will . . . be (larger, equal to, exactly twice, ...) compared to . . . ?”
- “When will . . . be as big (as small) as possible?”

- **Generalize/make conjectures:**

- “When does . . . work?” “Under what conditions does ... behave this way?”
- “Describe how to find . . . ?” “Is this always true?”

- **Justify/prove mathematically:**

- “Why does . . . work?” Will it always work?

- **Consider assumptions inherent in the problem:**

- “What will happen if the assumptions are changed?”
- For example, “suppose the triangle is not a right triangle, then....?”

- **Interpret information, make/ justify conclusions:**

- “The data support... ; “This... will makehappen because...”

We offer two observations about what good inquiry questions are *not*:

Good inquiry questions are not determined by punctuation – a question mark at the end of a sentence does not mean that a student was asked to reflect on anything meaningful. For example, the question: “What did you get?” is in the form of a question, but it is simply a prompt for the recording of an observation or result. “Explain how you can tell whether two columns of data are related by a linear equation” is not technically in the form of a question, but it requests the student to articulate some mathematical understanding. And of course, inquiry activity requiring sense making and reasoning may be based on the recording of results of previous performance activity, but it is critical that the sense be made by the student through questioning and reflection. Students only can learn what they actively make sense of, not passively receive what the teacher says.

Good inquiry questions aren't guaranteed simply by beginning them with one of the above stems. Thinking and conjecturing on the part of the students can be enhanced or inhibited depending on the kind of answers the questions elicit. Too often questions are simply requests for the recording of routine observations or procedural steps rather than invitations for students to make mathematical sense of a situation or to provide justification. Good inquiry questions can use the stems listed above, but they also need to be about the right knowledge types (*schematic* and *strategic* knowledge) embedded in the high-level mathematical task (*procedures with connections* or *doing mathematics*) being discussed.

Finally, we note that teachers need to be careful to provide feedback which is appropriate to the knowledge type which is the focus of the question. Simply telling a student, “Right! Good job!” or “No, you forgot to...” suggests that the question has a single right answer, as is usually the case with *declarative* and *procedural* knowledge. On the other hand, when the question focuses on *schematic* knowledge, appropriate feedback to a wrong answer might be something like, “think about what the equal sign means, and tell me why your statement makes sense.” When the question focuses on *strategic* knowledge, then appropriate feedback to a wrong answer might be something like, “if you're solving it that way, is that all you need to know?”

High Fidelity Learning Activities

We conclude this section by suggesting that the learning activities of *Algebra Nspired* and *Geometry Nspired* are designed to be “true” to the mathematics, to the learner, and to the teacher. These three dimensions of fidelity are described below:

- *Mathematical fidelity* - faithfulness of the representation’s properties and behavior to the mathematics. Some instances of mathematical infidelity may arise from modeling continuous phenomena with an inherently discrete machine. Authors may be unable to avoid completely such inherent limitations, but they have been mindful of them in the construction of an action-consequence document. A particular concern is construction of linkages that misportray the mathematics. For example, the notion of slope as a rate of change is a characteristic of a line in a coordinate plane that depends on directed changes in coordinates; to interpret it as ratio of segment lengths in a geometric world violates the mathematical integrity of the concept. In another example, notations and terminology used throughout the learning action-consequence document and the accompanying learning activity must also adhere carefully to accepted mathematical conventions.

- *Cognitive fidelity* - faithfulness of the representation’s properties and behavior to the perception of the user. If the external representations afforded by the action-consequence document are meant to support the mental representations of the learner, then the representation should not introduce cognitive obstacles for the learner. Consequently, authors carefully consider the potential for students to attend to *non-meaningful* cues or consequences, as well as the clarity of meaningful ones. Three examples can illustrate this point: first, a screen that is cluttered with irrelevant detail (perhaps to add “context” or “interest value,”) or which shows an unnecessary number of decimals in measurements, may distract from or confuse the student’s ability to attend to the important cues and consequences of their actions. Second, when using a slider bar to help students understand the parameter a in graphing a function such as $y = \sin(ax)$, careful highlighting is needed to direct attention.

Otherwise, a student's might attend more to the motion of the slider than to the manipulation of the values for a , and thus miss making the desired connection to the graph. Third, inattention to scaling can create situations where the underlying mathematics may be technically correct, but the visual perception of the user suggests an error (for example, perpendicular lines appearing otherwise).

- *Pedagogical fidelity* - faithfulness to the instructional goals of the teacher. The technological tool (in this case, TI-Nspire) needs to be clearly seen by the teacher as allowing students to engage in mathematical activity that the teachers believe will support student learning. Action-consequence documents can provide the basis for formulating effective questions: ones that push student thinking forward and/or provide valuable feedback to the teacher on how students are thinking. The examples in the discussion of Principle 4 are particularly relevant here.

3. The Math Nspired Lesson Resource Center

Our goal in designing the *Math Nspired* lesson resource center (currently including *Algebra Nspired* and *Geometry Nspired*) is to create a series of convenient, quick- and easy-to-use, self-contained, rich learning activities with embedded professional development, by applying the four principles above (see <http://www.algebranspired.com>).

These learning activities supplement, rather than supplant, the curriculum available in current textbooks. By focusing on the important math, they are universal in their applicability to almost any curriculum. The value they bring to the curriculum surrounds their use of dynamic visualization (through the action-consequence principle) to teach the important math underlying tough-to-teach/tough-to-learn topics. Their benefit is to help both teachers and students engage directly with the math, with a minimum of attention to the details of operating the technology.

The learning activities are intended to be useful “as is” to a teacher who is new to teaching, the technology, or the curriculum, with a minimum of preparation time. Experienced teachers and curriculum leaders will appreciate that the learning activities are open for customization, extension and adaptation to local needs. The activities include suggestions for how to use them in both large- and small-group classes. They do not presume a particular student ability level or classroom management plan.

Components of the System

Each *Math Nspired* lesson consists of three types of components:

- A learning activity, such as the *action-consequence document* (see Principle #3, above), in the form of a .TNS file ready to be loaded onto TI-Nspire handhelds or software.
- *Teacher Notes* -A supporting lesson plan, with questions consistent with the reflection and sense-making Principle #4, above. The notes also include Teacher Tips, for embedded professional development discussing the pedagogical content knowledge and pedagogy of specific points in each lesson. An example is below.

- *Student Activity* worksheets accompanying the lesson. These are included in both Adobe Acrobat (PDF) and Microsoft Word (DOC) format, for adaptation as desired.

While action/consequence lessons are the main focus, complimentary lessons are also included in *Math Nspired*, including “create your own” lessons that show educators step-by-step how to create TI-Nspire documents and exemplary lessons from TI-Math.com that extend teachers knowledge of how to use TI-Nspire technology with their students.

All lessons are packaged in a single, downloadable WinZip file (.ZIP), and they are available to educators on a single Web site the *Math Nspired* lesson resource center. In addition to the lessons, the Web site has many additional resources for teachers, including just-in-time video tutorials, a getting started guide, and alignment to curriculum standards, assessment standards and algebra/geometry textbooks.

In addition, customized blended professional development opportunities are available to help math leaders and their teachers effectively use the *Nspired* resources in their classrooms, with their students.

The *Math Nspired* series of curriculum resources are designed to help teachers and students get to the important math as quickly and easily as possible. Each learning activity brings together principles drawn from research on math and pedagogical math content, cognitive psychology, technology, and pedagogy. The resources are designed to give teachers new approaches to teaching the important, tough-to-teach/tough-to-learn mathematics which are at the root of many of the most pervasive barriers to success on high-stakes tests faced by the majority of students. They use the four design principles discussed in this paper to create instructional solutions with high fidelity to the math, the cognitive level of knowledge, and the pedagogy needed to teach that knowledge effectively. They scaffold both teacher and student, to facilitate dialog on math tasks with high cognitive demand, which emphasize schematic and strategic knowledge, while using declarative and procedural knowledge embedded in each curriculum topic. They use technology in a way which will be new to many teachers, to place students and teachers in an exploratory microworld carefully designed to show the critical attributes and relationships of the schematic knowledge involved.

The Math Nspired curriculum resource series also reflects current research on scalability of technological innovations in education: how to create innovative curriculum resources which will be used very widely, by new and experienced teachers who seek a practical tool they can quickly and easily integrate into their teaching. With a minimum of conventional professional development, teachers at any experience level should be able to quickly learn to use the lessons to get to the math.

An Example from Algebra Nspired

Here is an excerpt of the lesson plan which accompanies the action-consequence document described in the opening example.

Math Objectives:

- To understand what it means for an ordered pair to be a solution to a linear equation
- To understand what it means for an ordered pair to be a solution to a system of linear equations

About the Math:

This lesson involves **solving a system of linear equations**. The emphasis is on helping students understand that the solution to a system of equations is an ordered pair of numbers that makes both equations true (or “balances” both equations) at the same time. By using this visual and numerical approach to solving the system, students will discover the meaning of a numerical solution to a system of equations. Students can explore systems of linear equations that have infinitely many or no solutions, non-linear systems, as well as systems with non-integer and negative solutions. A good precursor activity might be to use the activity *What is A Variable* where students investigate how the value of an expression of the form $mx+b$ for a given m and b changes for a given change in x . This way of reasoning can be very helpful in figuring out how to work with the balance.

In the table below, the student worksheet directions are on the left. On the right are comments which explain to readers of this paper the design of the learning activity.

The teacher’s version of the actual worksheet includes the *Teacher Tips*, shown in the right-hand column in *italics*.

<u>Lesson Worksheet</u>	<u>Comments</u>
<p>1. Move the arrows until $x = 3$ and $y = 6$.</p> <p>a. Describe what each scale looks like.</p> <p>b. Why are they in this position?</p> <p>Problems 2 and 3 are intended to probe student understanding of what it means to be a solution to a linear equation in two unknowns.</p> <p>2. a, What does it mean if a scale is <i>balanced</i>?</p> <p>b. If $x = -5$, what value of y will balance the left scale? The right scale?</p> <p>3. Find a point such that the scale on the left is balanced.</p> <p>a. Describe what the balance tells you about the size of the two expressions on either side of the scale on the right.</p> <p>b. Should there be other points that will balance the scale on the left? Why or why not?</p>	<p>The first 3 problems are intended to familiarize the student with the action and consequence. Here the drivers are the two points on the number line and the consequence is a shift in the balance of the two equations.</p>

<u>Lesson Worksheet</u>	<u>Comments</u>																								
<p>4. . Find three ordered pairs, (x,y), that balance the left scale. Describe the strategy you used to find these points.</p> <p>5. Find three ordered pairs, (x,y), that balance the right scale. How was your strategy for this problem the same or different from the one you used in problem 4?</p> <p>6. Find values for x and y that satisfy the conditions in the table below.</p>	<p>Problems 4 and 5 are intended to push students into thinking about strategies for balancing the scales rather than just moving the points at random.</p> <p>Teacher Tip: Again have students share and if their strategies are different, ask them to explain why they choose another method for this problem. They should notice that as x increases by 1, y has to decrease by 1, which ties it to the concept of slope or rate of change and the fact that the rate of change is constant. Reinforce that these ordered pairs are all solutions to the equation $x + y = 9$. You may want to reinforce the fact that there are an infinite number of solutions to this equation as well.</p>																								
<table border="1"> <thead> <tr> <th colspan="2" data-bbox="237 1087 493 1178">Values</th> <th colspan="2" data-bbox="493 1087 797 1178">Is It Balanced?</th> </tr> <tr> <th data-bbox="237 1178 367 1318">x</th> <th data-bbox="367 1178 493 1318">y</th> <th data-bbox="493 1178 639 1318">Left Side</th> <th data-bbox="639 1178 797 1318">Right Side</th> </tr> </thead> <tbody> <tr> <td></td> <td></td> <td data-bbox="493 1318 639 1409">Yes</td> <td data-bbox="639 1318 797 1409">No</td> </tr> <tr> <td></td> <td></td> <td data-bbox="493 1409 639 1499">No</td> <td data-bbox="639 1409 797 1499">Yes</td> </tr> <tr> <td></td> <td></td> <td data-bbox="493 1499 639 1589">No</td> <td data-bbox="639 1499 797 1589">No</td> </tr> <tr> <td></td> <td></td> <td data-bbox="493 1589 639 1680">Yes</td> <td data-bbox="639 1589 797 1680">Yes</td> </tr> </tbody> </table>	Values		Is It Balanced?		x	y	Left Side	Right Side			Yes	No			No	Yes			No	No			Yes	Yes	<p>Problems 6, 7 and 8 are the core problems that lead to the objective of the lesson, finding and interpreting the solution to a system of two linear equations.</p> <p>Teacher Tip: This is an important time to have a class discussion. Ask, “How many of you had the same pair in the first row? Second row? Third? Fourth?” The first three rows may or may not be the same; however, the 4th row must be the same. It is important for students to recognize there are an infinite number of solutions for each of the individual equations, however there is only one solution that satisfies both equations at the same time. This is called the solution to the system. Have students share how they found their solution for the fourth row and ask the class if they think</p>
Values		Is It Balanced?																							
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<u>Lesson Worksheet</u>	<u>Comments</u>
<p>7. Compare your table from #5 with a neighbor's table. Were any of your answers the same? If so, which ones? Discuss why some of them might be the same and some might be different.</p> <p>8. What is the significance of the last row of the table in problem 5?</p> <p>Move to page 1.3 of the TI-Nspire document</p> <p>9. How many solutions are there for this system of equations? How do you know?</p> <p>Move to page 1.4 of the TI-Nspire document.</p> <p>10. How many solutions are there for this system of equations? How do you know?</p>	<p><i>the strategies would work for another set of equations and why or why not.</i></p> <p>Problems 9 and 10 ask students to apply what they learned thus far in two new contexts, where the systems are different in nature (one is dependent and the other inconsistent), but the reasoning process they have used should enable them to make sense of these new situations.</p> <p>Teacher Tip: <i>It is critical that students end this activity by comparing the three different situations: sometimes every ordered pair that balances one balances the other, sometimes no ordered pair balances them both and sometimes only one ordered pair balances them both. What might you notice about the equations that would help you figure this out? Even if the ideas are not yet firm, students should begin to see that the notion of same relationship between the coefficients of x and y have something to do with these different situations, and that the notion of having the same or different rate of change as you go from one ordered pair that balances a scale to another that also</i></p>

<u>Lesson Worksheet</u>	<u>Comments</u>
	<i>balances the scale is key. This leads to the activity, How Many Solutions to a System?</i>

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